Thoughts on Tree Construction

The diagram below shows the unrooted binary tree possessing 3 leaf nodes (labeled S₁, S₂, S₃) and one internal node (labeled A) of degree 3. Note that there is only one such tree; rotation of the leaves does not change its fundamental structure (i.e. the “distance” in hops between the leaf nodes does not change).

How to construct the unrooted trees with 4 leaf nodes? Clearly, we need to add one new edge for the new leaf node S₄, and there are 3 current edges to which it can be attached, creating new internal node B. The diagram below shows the 3 resulting trees.

To construct the trees with 5 leaf nodes, we add an edge to each of the 5 edges in the 3 trees above. Following this logic of tree construction, there is (1) unrooted binary tree with 3 leaf nodes, (1)(3) = 3 trees with 4 leaf nodes, (1)(3)(5) = 15 trees with 5 leaf nodes, etc. Therefore, there are \( \prod_{i=3}^{n} (2i-5) \) unrooted binary trees with \( n \) leaf nodes.

We construct the rooted trees with a similar construction method. Simply add an edge from the root to each edge in the unrooted tree. Below are the 3 rooted trees with 3 leaf nodes, based on the unrooted 3-leaf node tree above.

Similarly, here are a few of the 15 rooted trees with 4 leaf nodes. They are constructed based on the leftmost of the 3-leaf node trees seen above.

There are (1)(3) = 3 rooted binary trees with 3 leaf nodes, (1)(3)(5) = 15 trees with 4 leaf nodes, etc.. Therefore, there are \( \prod_{i=3}^{n} (2i-3) \) rooted binary trees with \( n \) leaf nodes.