MTH 225(M)

Name: last, first (Name: last, first) Feb:27(Fri)

(You are allowed to work in teams of two, but there is no teamwork bonus for this test.)

# Midterm Examination

## Due by 2:00 p.m. March 11(Wed)

*Acknowledgement: Some of these exercises are derived from Kenneth Rosen.*

A. For each of the following English sentences, explain very briefly which kind of "or" is intended, an inclusive "or" (that is, a disjunction) versus an exclusive "or":

1. [2 points] "When you buy a new car from Acme Motor Company, you get $2000 back in cash or a 2% car loan."

2. [2 points] "School is closed if more than 2 feet of snow falls or if the wind chill is below -100 ."

B. [2 points] Give the contrapositive of the following implication:

"If it is raining today, then I am driving to work."

C. 1. [6 points] Complete a truth table as follows for the logic formula $\texttt{P} \vee (\neg\texttt{Q} \to \texttt{R})$. Remember that you need to have a column for each subformula such as $\neg\texttt{Q}$ and $\neg\texttt{Q} \to \texttt{R}$ in addition to the columns for `P`, `Q`, `R`, and the final column for the overall formula.

```
 P | Q | R ||
---+---+---++-----------------------------------------------
 t | t | t ||
---+---+---++-----------------------------------------------
 t | t | f ||
---+---+---++-----------------------------------------------
 t | f | t ||
---+---+---++-----------------------------------------------
 t | f | f ||
---+---+---++-----------------------------------------------
 f | t | t ||
---+---+---++-----------------------------------------------
 f | t | f ||
---+---+---++-----------------------------------------------
 f | f | t ||
---+---+---++-----------------------------------------------
 f | f | f ||
```

2. [2 points] Is the logic formula $\texttt{P} \vee (\neg\texttt{Q} \rightarrow \texttt{R})$ a tautology? Explain.

D. [8 points] Let R(x) represent the statement "x can speak Russian" and let C(x) represent the statement "x knows the computer language C++". Express each of the following sentences in terms of "R(x)", "C(x)", quantifiers, and logical connectives. The domain for quantifiers consists of all students at your school.

- There is a student at your school who can speak Russian but who doesn't know C++.
- Every student at your school either can speak Russian or knows C++.
- There are two students at your school who are different people who know Russian.
- There is one particular student at your school such that for each student at your school, the latter student knows both Russian and C++ if and only if the latter student is actually the same person as that one particular student.

E. [10 points] Let P(x) represent the statement "$x < x^2$". What are the truth values of the following formulas?

1. $\texttt{P}(-2)$
2. $\texttt{P}(1)$
3. $\texttt{P}(3)$
4. $\exists x \in \mathbb{N}\, [\texttt{P}(x)]$
   Explain.
5. $\forall x \in \mathbb{N}\, [\texttt{P}(x)]$
   Explain.
6. $\exists x \in \mathbb{N}\, [\neg\texttt{P}(x)]$
   Explain.
7. $\forall x \in \mathbb{N}\, [\neg\texttt{P}(x)]$
   Explain.

F. [6 points] Here are some of the logical simplifications. A "$\varphi$" (possibly subscripted) represents an arbitrary logic formula, "f" stands for "`false`", and "t" stands for "`true`".

$$
\begin{array}{cccccccc}
\mathtt{f} \land \varphi & \varphi \land \mathtt{f} & \mathtt{t} \land \varphi & \varphi \land \mathtt{t} & \varphi \land \varphi & \varphi \land \neg\varphi & \neg\varphi \land \varphi & \\
\Downarrow & \Downarrow & \Downarrow & \Downarrow & \Downarrow & \Downarrow & \Downarrow & \\
\mathtt{f} & \mathtt{f} & \varphi & \varphi & \varphi & \mathtt{f} & \mathtt{f} & \\
\mathtt{f} \lor \varphi & \varphi \lor \mathtt{f} & \mathtt{t} \lor \varphi & \varphi \lor \mathtt{t} & \varphi \lor \varphi & \varphi \lor \neg\varphi & \neg\varphi \lor \varphi & \\
\Downarrow & \Downarrow & \Downarrow & \Downarrow & \Downarrow & \Downarrow & \Downarrow & \\
\varphi & \varphi & \mathtt{t} & \mathtt{t} & \varphi & \mathtt{t} & \mathtt{t} & \\
\mathtt{f} \rightarrow \varphi & \varphi \rightarrow \mathtt{f} & \mathtt{t} \rightarrow \varphi & \varphi \rightarrow \mathtt{t} & \varphi \rightarrow \varphi & \varphi \rightarrow \neg\varphi & \neg\varphi \rightarrow \varphi & \neg\varphi_1 \rightarrow \neg\varphi_2 \\
\Downarrow & \Downarrow & \Downarrow & \Downarrow & \Downarrow & \Downarrow & \Downarrow & \Downarrow \\
\mathtt{t} & \neg\varphi & \varphi & \mathtt{t} & \mathtt{t} & \varphi & \neg\varphi & \varphi_2 \rightarrow \varphi_1
\end{array}
$$

$$
\begin{array}{cccc}
\neg\neg\varphi & \neg(\neg\varphi_1 \land \neg\varphi_2) & \neg(\neg\varphi_1 \lor \neg\varphi_2) & \neg(\varphi_1 \rightarrow \neg\varphi_2) \\
\Downarrow & \Downarrow & \Downarrow & \Downarrow \\
\varphi & \varphi_1 \lor \varphi_2 & \varphi_1 \land \varphi_2 & \varphi_1 \land \varphi_2
\end{array}
$$

Use those simplifications — *and no other transformations* — to simplify the following formula as much as possible. (Note: your result here may be different from the result you could get from `ProofBuilder`.) Show the steps of simplification, including drawing a box around each subpart before you simplify it and also circling the simplification above that you use. (Don't worry about trying to show exactly when you use each simplification: I'll be able to figure it out.) You can do multiple simplifications at one time, but only if the multiple boxes that you need to draw for them do not overlap. And again, use the simplifications listed above, *NO OTHER TRANSFORMATIONS!*

$$\neg[\neg\mathtt{P} \land (\mathtt{true} \rightarrow \neg\mathtt{Q})\ ] \ \lor \ (\mathtt{R} \rightarrow \mathtt{false})$$

G. [3 points] Consider the following Java code:

```
if ( !((d < n*m) || (f(d-n*m) != 1)) )
    System.out.println("hello");
```

Give an expression that is equivalent to the conditional expression there but which contains no occurrences of Java's logical negation operator "`!`". In doing this exercise, you may want to try to remember De Morgan's laws. (But it's fine if you manage to do this exercise without remembering them: you don't need to show them or anything; I'm just suggesting that they could be useful here.)

H. [12 points] Let the universe of discourse be the set $\mathtt{U} := \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Then let $\mathtt{A} := \{2, 4, 8\}$, $\mathtt{B} := \{1, 3, 6, 9\}$, and $\mathtt{C} := \{6, 7, 8, 9\}$.

1. Give a concise list representation of each of the sets produced by the following operations (i.e., explicitly list the elements with braces, or write just "∅" if appropriate); and also give the cardinality of each of the sets that are produced: cardinality:

   $\mathtt{A} \cap \mathtt{C}$ :=

   $\mathtt{B} \cap \mathtt{C}$ :=

   $(\mathtt{A} \cap \mathtt{C}) \cup (\mathtt{B} \cap \mathtt{C})$ :=

   $\mathtt{A} \cup \mathtt{B}$ :=

   $(\mathtt{A} \cup \mathtt{B}) \cap \mathtt{C}$ :=

   $\mathtt{B} - \mathtt{C}$ :=

   $\overline{\mathtt{A}}$ a.k.a. $\sim\mathtt{A}$ :=

   $\mathtt{C} \cup \varnothing$ :=

   $\mathtt{A} \cap \varnothing$ :=

2. Look at $(\mathtt{A} \cap \mathtt{C}) \cup (\mathtt{B} \cap \mathtt{C})$ and $(\mathtt{A} \cup \mathtt{B}) \cap \mathtt{C}$ above. For any sets $\mathtt{S}_1$, $\mathtt{S}_2$, and $\mathtt{S}_3$, what should be the relationship between $(\mathtt{S}_1 \cap \mathtt{S}_3) \cup (\mathtt{S}_2 \cap \mathtt{S}_3)$ and $(\mathtt{S}_1 \cup \mathtt{S}_2) \cap \mathtt{S}_3$ ?

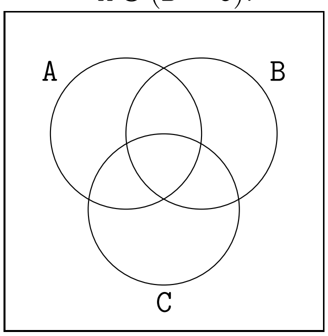
3. Which of the pairs of sets $(\mathtt{A}, \mathtt{B})$, $(\mathtt{A}, \mathtt{C})$, $(\mathtt{B}, \mathtt{C})$ are disjoint?

I. [4 points] Give concise list representations of the sets that are specified as follows (explicitly list the elements with braces, or write just "∅" if appropriate):

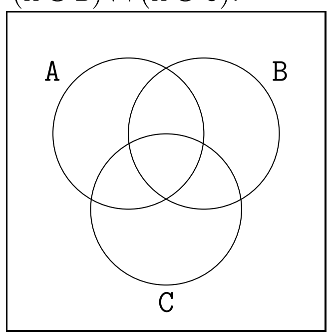
- $\{\, x : x \text{ is an integer and } x^2 = 2 \,\}$
- $\{\, x : x \text{ is a real number and } x^2 = 4 \,\}$
- $\{\, x : x \text{ is the cube of a nonnegative integer and } x < 100 \,\}$
- $\{\, S : S \text{ is the set of letters of either your first name or your middle name or your last name} \,\}$
  Use lowercase for every letter.

J. [6 points] Shade the following Venn diagrams to match the following indicated combinations of three sets A, B, and C:

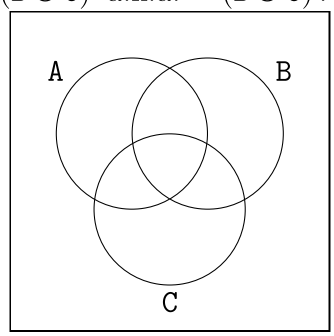
$A \cup (B - C)$:




$(A \cup B) \cap (A \cup C)$:




$\overline{(B \cup C)}$ a.k.a. $\sim(B \cup C)$ :




K. [6 points] Let A be the set of English words that contain the letter "x", and let B be the set of English words that contain the letter "q". Express each of the following sets as a combination of A and B (using set operations).

1. The set of English words that do not contain the letter "x":
2. The set of English words that contain both an "x" and a "q":
3. The set of English words that contain an "x" but not a "q":
4. The set of English words that do not contain either an "x" or a "q", or to put it another way, that contain neither an "x" nor a "q",
5. The set of English words that contain either an "x" or a "q", but not both:

L. [3 points] Determine the contents of the sets A and B if $A - B := \{1, 5, 7, 8\}$, $B - A := \{2, 10\}$, and $A \cap B := \{3, 6, 9\}$:

$A :=$

$B :=$

M. [4 points] Give the power set indicated as follows:

$\mathcal{P}(\{a, b\}) :=$

N. [3 points] Let $A := \{r, g, b\}$ and $B := \{0, 1\}$. Give $A \times B$ :

O. [9 points]  Redo the work of Exercise E above using `ProofBuilder`. But for this exercise, you don't need to use the "P()" notation: for examples you can enter more simply $-2 < (-2)^2$, $1 < 1^2$, $\forall x{\in}\mathbb{N}\,[x < x^2]$, etc. — except remember to enter "¬" preceding each formula that is false (e.g. if the formula were "$0 < 1$" then you could enter it straightforwardly, but if the formula were "$1 < 0$" then you would enter "$\neg[1 < 0]$").

Rem. you can copy material from documents such as this one's PDF which is online and paste such into `ProofBuilder`. (You may need to fix some glitches.)

P. [3 points]  Starting with the following formula, in `ProofBuilder` apply equivalences to produce an equivalent formula where negations are applied to only predicates (that is, where no negation is outside a quantifier or an expression involving logical connectives).

$$\neg\forall y\exists x[P(x,y) \vee Q(x,y)]$$

Rem. you can copy material from documents such as this one's PDF which is online and paste such into `ProofBuilder`. (You may need to fix some glitches.)

Q. [8 points]  Here are some set axioms:

$$\forall S_1\,\forall S_2\,[S_1 = S_2 \;\leftrightarrow\; \forall x\,(x \in S_1 \;\leftrightarrow\; x \in S_2)]$$
$$\forall S_1\,\forall S_2\,\forall x\,[x \in S_1 - S_2 \;\leftrightarrow\; (x \in S_1 \;\wedge\; x \notin S_2)]$$
$$\forall S\,\forall x\,[x \in {\sim}S \;\leftrightarrow\; x \notin S)]$$
$$\forall S_1\,\forall S_2\,\forall x\,[x \in S_1 \cap S_2 \;\leftrightarrow\; (x \in S_1 \;\wedge\; x \in S_2)]$$

Use those axioms as presuppositions in `ProofBuilder` to prove the following formula:

$$\forall S_1\,\forall S_2\,[S_1 - S_2 \;=\; S_1 \cap {\sim}S_2]$$

Rem. you can copy material from documents such as this one's PDF which is online and paste such into `ProofBuilder`. (You may need to fix some glitches.)

Total Score: / 99