Name: last, first

MTH 225-02
December:09(Tue)

Final Examination

- During this test, you should not use any auxiliary materials or computational devices; i.e. this test is 'closed-book' (and 'closed-notes' etc.). But if you can't remember some little detail, ask the instructor.
- Write legibly. You should use ink (i.e. a pen) rather than pencil.
- Show intermediate work / explanations for obtaining answers.

Acknowledgement: Some of these exercises are derived from Kenneth Rosen.

A. [4 points] Demonstrate using simply subtraction (and counting) — not division nor multiplication nor addition nor other operations — to determine the value of the expression `x mod y` and the value of the expression $\lfloor \frac{\mathtt{x}}{\mathtt{y}} \rfloor$ when `x` is 174 and `y` is 27 (you may not need to use all of the following material): 174 -27| -27| -27| -27| -27| -27| -27| -27| -27|

By the way, what is the value of the expression $\lceil \frac{174}{27} \rceil$

B. [7 points]

- Show the result as well as the sequence of values that the parameters `x` and `y` have during performance of the Euclidean algorithm for the invocation `gcd(549,156)`. For example, for `gcd(90,60): 90, 60, 30, 0` — plus the result would be needed.

- What is `549/gcd(549,156)`?
- What is `154/gcd(549,156)`?
- Are `549/gcd(549,156)` and `156/gcd(549,156)` relatively prime?

C. [4 points] Find integers `x` and `y` so that `17x + 21y := 1`. (Try values between `-6` and `6`.) Then, using the values of `x` and `y` that you obtain here, complete the following chart, giving the specified values:

```
  x  |  y  |  17x  |  21y  |  17x mod 21
-----+-----+-------+-------+------------
     |     |       |       |
     |     |       |       |
```

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D. [7 points] Consider that $43 \cdot (-5) + 72 \cdot 3 := 1$.

- Considering that, give an integer value d between 1 and 71 that is a multiplicative inverse of 43 modulo 72: $d :=$
- Give the values of the primes p and q for which $p \cdot q := 91$:
- Give the formula for $\phi(z)$ when $z := p \cdot q$ (with p and q being primes):
- Then give the value of $\phi(91)$:
- Use a property of exponentiation "x^y" to give an expression equivalent to $(55^{43})^d$ without parentheses. (Your answer here should still contain the symbol "d"; you should not do any arithmetic here with the value for d that you determined above, you just need to come up with the equivalent expression.):
- The value of the expression $18^{43} \bmod 91$ happens to be 60. Considering that and things above, using the value for d from above, what do you think is the value of $60^d \bmod 91$?
- Where is this mathematical material used in real Computer Science and Information Systems?

E. [7 points] Let the universe of discourse be the set $U := \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Then let $A := \{1, 2, 4, 8\}$, $B := \{1, 3, 6, 9\}$, and $C := \{5, 6, 7\}$.

1. Give list representations of the sets produced by the following operations (i.e., explicitly list the elements with braces, or write just "$\emptyset$" if appropriate), and also give the cardinality of each of the sets that you obtain here: <u>cardinality</u>

   $A \cap B :=$

   $B - C :=$

   A^c a.k.a. $\overline{A} :=$

   $C \cup \emptyset :=$

   $A \cap \emptyset :=$

2. Which of the pairs of sets (A, B), (A, C), (B, C) are disjoint?

F. [10 points] The symbol "$\prod$" means "product" like the symbol "$\sum$" means "summation".

Determine the value of $\prod_{0 \le i \le 3} i^2$:

***Ask** if you want help with this.*

Determine the value of $\prod_{0 \le i \le 4} i^2$:

For every natural number (i.e. nonnegative integer) n, what is the value of $\prod_{0 \le i \le n} i^2$? Let "Z" denote this value; $Z :=$

Use quantifiers and logical connectives etc. to express the following statement:

"For every n, the value of $\prod_{0 \le i \le n} i^2$ is Z."

Use quantifiers and logical connectives etc. to express the following statement:

"There is a value v such that for every n, the value of $\prod_{0 \le i \le n} i^2$ is v."

Determine the value of $\prod_{1 \le i \le 3} i^2$:

Determine the value of $\prod_{1 \le i \le 4} i^2$:

Use quantifiers and logical connectives etc. to express the following statement:

"There isn't a value v such that for every n, the value of $\prod_{1 \le i \le n} i^2$ is v."

Use quantifiers and logical connectives etc. to express the following statement:

"There are values m and n such that m and n are different and the values of $\prod_{1 \le i \le m} i^2$ and of $\prod_{1 \le i \le n} i^2$ are different."

Give a simple formula for $\prod_{1 \le i \le n} i^2$:

(Hint: What is the simple formula for $\prod_{1 \le i \le n} i$? Or if you want, I can give you another hint for this.)

G. [3 points] Suppose 25 teams compete in an event where the best 2 teams are considered the winning pair. *(ACM regional programming contests are like this.)*

- Then determine how many different winning pairs there might possibly be:

- And then further if GVSU is one of the 25 competing teams, determine how many of all those possible winning pairs have GVSU as one of the two winners:

H. [2 points] What is the largest integer n for which we can be confident that in any group of 100 people, at least n of them have their birthdays in the same month? Use the (generalized) Pigeonhole Principle.

I. 1. [6 points] Complete a truth table as follows for the logic formula $\mathtt{P} \lor (\neg\mathtt{Q} \Rightarrow \mathtt{R})$. Remember that you need to have a column for each subformula such as $\neg\mathtt{Q}$ and $\neg\mathtt{Q} \Rightarrow \mathtt{R}$ in addition to the columns for P, Q, R, and the final column for the overall formula.

P	Q	R	
t	t	t	
t	t	f	
t	f	t	
t	f	f	
f	t	t	
f	t	f	
f	f	t	
f	f	f	

2. [2 points] Is the logic formula $\mathtt{P} \lor (\neg\mathtt{Q} \Rightarrow \mathtt{R})$. a tautology? Explain.

J. [6 points] Here are some of the logical simplifications. The symbol "φ" (possibly subscripted) represents any arbitrary logic formula.

$\mathtt{f} \land \varphi \longrightarrow \mathtt{f}$	$\varphi \land \mathtt{f} \longrightarrow \mathtt{f}$	$\mathtt{t} \land \varphi \longrightarrow \varphi$	$\varphi \land \mathtt{t} \longrightarrow \varphi$
$\varphi \land \varphi \longrightarrow \varphi$	$\varphi \land \neg\varphi \longrightarrow \mathtt{f}$	$\neg\varphi \land \varphi \longrightarrow \mathtt{f}$	
$\mathtt{f} \lor \varphi \longrightarrow \varphi$	$\varphi \lor \mathtt{f} \longrightarrow \varphi$	$\mathtt{t} \lor \varphi \longrightarrow \mathtt{t}$	$\varphi \lor \mathtt{t} \longrightarrow \mathtt{t}$
$\varphi \lor \varphi \longrightarrow \varphi$	$\varphi \lor \neg\varphi \longrightarrow \mathtt{t}$	$\neg\varphi \lor \varphi \longrightarrow \mathtt{t}$	
$\mathtt{f} \Rightarrow \varphi \longrightarrow \mathtt{t}$	$\varphi \Rightarrow \mathtt{f} \longrightarrow \neg\varphi$	$\mathtt{t} \Rightarrow \varphi \longrightarrow \varphi$	$\varphi \Rightarrow \mathtt{t} \longrightarrow \mathtt{t}$
$\varphi \Rightarrow \varphi \longrightarrow \mathtt{t}$	$\varphi \Rightarrow \neg\varphi \longrightarrow \varphi$	$\neg\varphi \Rightarrow \varphi \longrightarrow \neg\varphi$	
$\neg\varphi_1 \Rightarrow \neg\varphi_2 \longrightarrow \varphi_2 \Rightarrow \varphi_1$	$\neg(\varphi_1 \Rightarrow \neg\varphi_2) \longrightarrow \varphi_1 \land \varphi_2$	$\neg\neg\varphi \longrightarrow \varphi$	
$\neg(\neg\varphi_1 \land \neg\varphi_2) \longrightarrow \varphi_1 \lor \varphi_2$	$\neg(\neg\varphi_1 \lor \neg\varphi_2) \longrightarrow \varphi_1 \land \varphi_2$		

Use those simplifications — *and no other transformations* — to simplify the following formula as much as possible. Show the steps of simplification, including drawing a box around each subpart before you simplify it and also circling the simplifications above that you use. (Don't worry about trying to show exactly when you use them: I'll be able to figure it out.) You can do multiple simplifications at one time, but only if the multiple boxes that you need to draw for them do not overlap. Again, use the simplifications listed above, **no other transformations**.

$$[\neg\mathtt{P} \lor (\mathtt{t} \land \neg\mathtt{Q})\ ] \ \Rightarrow\ (\mathtt{R} \land \mathtt{f})$$

K. [7 points] Write a proof by induction of the following formula: $(\forall \mathtt{n} \in \mathbb{N})[\mathtt{n} < 2^{\mathtt{n}}]$. In your proof, you can use the following fact: $(\forall \mathtt{x} \in \mathbb{N})[\mathtt{1} < 2^{\mathtt{x}}]$